

Remark About Hamiltonian Formulation of Non-Linear Massive Gravity in Stückelberg Formalism

J. Klusoň

*Department of Theoretical Physics and Astrophysics
Faculty of Science, Masaryk University
Kotlářská 2, 611 37, Brno
Czech Republic
E-mail: klu@physics.muni.cz*

ABSTRACT: We perform the Hamiltonian analysis of the specific model of the non-linear massive gravity in Stückelberg formalism where the square root structure is replaced by introducing auxiliary fields. We show that the constraint structure of given theory varies over the phase space and discuss possible consequences of this fact for the counting of the physical degrees of freedom.

KEYWORDS: Massive Gravity.

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1. Introduction and Summary

One of the most challenging problem is to find consistent formulation of massive gravity. The first attempt for construction of this theory is dated to the year 1939 when Fierz and Pauli formulated its version of linear massive gravity [1]¹. However it is very non-trivial task to find a consistent non-linear generalization of given theory and it remains as an intriguing theoretical problem. It is also important to stress that recent discovery of dark energy and associated cosmological constant problem has prompted investigations in the long distance modifications of general relativity, for review, see [3].

It is natural to ask the question whether it is possible to construct theory of massive gravity where one of the constraint equation and associated secondary constraint eliminates the propagating scalar mode. It is remarkable that linear Fierz-Pauli theory does not suffer from the presence of such a ghost. On the other hand it was shown by Boulware and Deser [5] that ghosts generically reappear at the non-linear level. On the other hand it was shown recently by de Rham and Gobadadze in [8] that it is possible to find such a formulation of the massive gravity which is ghost free in the decoupling limit. Then it was shown in [9] that this action that was written in the perturbative form can be resummed into fully non-linear actions². The general analysis of the constraints of given theory has been performed in [6]. It was argued there that it is possible to perform such a redefinition of the shift function so that the resulting theory still contains the Hamiltonian constraint. Then it was argued that the presence of this constraint allows to eliminate the scalar mode and hence the resulting theory is the ghost free massive gravity. However this analysis was questioned in [36] where it was argued that it is possible that this constraint is the second class constraint so that the phase space of given theory would be odd dimensional. On the other hand in the recent paper [37] very nice analysis of the Hamiltonian formulation of the most general gauge fixed non-linear massive gravity actions was performed with an important conclusion that the Hamiltonian constraints has zero Poisson brackets. Then the requirement of the preservation of this constraint during the time evolution of the

¹For review, see [2].

²For related works, see [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34].

system implies an additional constraint. As a result given theory has the right number of constraints for the construction of non-linear massive gravity without additional scalar mode ³.

All these results suggest that the gauge fixed form of the non-linear massive gravity actions could be ghost free theory. On the other hand the manifest diffeomorphism invariance is lacking and one would like to confirm the same result in the gauge invariant formulation of the massive gravity action using the Stückelberg fields. In fact, it was argued in [35] that for some special cases such a theory possesses an additional primary constraint whose presence implies such a constraint structure of given theory that could eliminate one additional scalar mode. The generalization of given work to the case general metric was performed [13]. We found the Hamiltonian form of given action and determined primary and secondary constraints of given theory. Then we showed that the algebra of the secondary constraints is closed and takes the form of the algebra of constraints of diffeomorphism invariant theory. Unfortunately we were not able to find an additional constraint that could lead to the elimination of the additional scalar mode with exception of the two dimensional case where however two dimensional gravity is trivial.

These results suggest that it is still very instructive to analyze the non-linear massive gravity further in order to fully understand to it. For that reason we perform the Hamiltonian analysis of the model of non-linear massive gravity action in the form that was presented in [38, 12]. We consider this action written in manifestly diffeomorphism invariant way using the collection of the Stückelberg fields. We perform the Hamiltonian analysis and we identify one important fact related to the symplectic structure of given theory. Namely we find that the rank of the matrix of the Poisson brackets between auxiliary fields varies over the phase space. Explicitly, for generic point of the phase space when the determinant of the matrix $V^{AB} = g^{ij}\partial_i\phi^A\partial_j\phi^B$ is non-zero we find that the theory possesses three spatial diffeomorphism constraints and one Hamiltonian constraint that are the first class constraints. Then the constraint structure of given theory at generic point of the phase space obeys the basic principles of geometrodynamics [39, 42, 43]. This result also implies that for these generic points there is no additional constraint that could eliminate one additional scalar mode and this fact agrees with the results derived in [13]. However an interesting situation occurs in the subspace of the phase space where $\det V^{AB} = 0$. In this case we find that the Poisson bracket between primary constraints $P_{AB} \approx 0$, where P_{AB} is momentum conjugate to the auxiliary field Φ^{AB} and corresponding secondary constraint Ψ_{AB} degenerates and possesses eigenvectors whose number is determined by the rank of the matrix V^{AB} . Then we perform the Hamiltonian analysis of the system in case when $\det V^{AB} = 0$ and we find that now there is one additional first class constraint given as the linear combinations of the constraints $P_{AB} \approx 0$. On the other hand we find that this secondary constraint is preserved during the time evolution of the system when the lapse function N which is the Lagrange multiplier of the Hamiltonian constraint vanishes. As a result the Hamiltonian constraint is the second class constraint. However we should stress one important difference between diffeomorphism invariant systems and the system where

³For related work, see [38].

the Hamiltonian is not given as the linear combination of the constraints. To see this let us imagine that the equation that determines the time evolution of given constraint reduces to the Poisson bracket between constraint and the Hamiltonian. Then in the case when the Hamiltonian does not vanish on the constraint surface the requirement of the preservation of given constraint would imply an additional constraint. On the other hand in case of the diffeomorphism invariant system we find that the Hamiltonian is given as the linear combination of the Hamiltonian constraint and spatial diffeomorphism constraints and hence the non-zero Poisson bracket between any constraint and Hamiltonian constraint would imply the equation which determines the value of the Lagrange multiplier which is the lapse N in this case. In summary we find that the number of the physical degrees of freedom defined on the phase space subspace $\det V^{AB} = 0$ is the same as the number of physical degrees of freedom determined for the generic point of the phase space.

At this point we should stress one important point. Systems where the Poisson brackets between constraints varies along the phase space are not well understood and it is possible that they possess very reach dynamics ⁴. It is not completely clear for us how to proceed with such systems and for that reason we should be very careful with giving some definitive conclusions. It would be certainly very interesting to study the phase space structure of given theory in more details, as for example in the paper [46]. Explicitly, it is not clear how to proceed with the theory on the degenerative surface $\det V^{AB} = 0$. Shall we interpret $\det V^{AB} = 0$ as an additional constraint that eliminate one additional degree of freedom? Such a condition can be imposed by the projectable condition $\phi^0 = \phi^0(t)$ which clearly does not correspond to any local degree of freedom. Further, it would be also very interesting to analyze the flow of the system near the surface $\det V^{AB} = 0$. We leave all these open questions for future.

The structure of given paper is as follows. In the next section (2) we give an overview of the non-linear massive gravity in the formulation presented in [38, 12]. We perform the Hamiltonian analysis of given theory and we identify the primary and the secondary constraints for the generic point of the phase space where $\det V^{AB} \neq 0$. Then in section (3) we analyze the theory on the phase space subspace $\det V = 0$ and discuss issues that are related to such systems.

2. Non-Linear Massive Gravity

Let us consider following non-linear massive gravity action [9]

$$S = M_p^2 \int d^4x \sqrt{-\hat{g}}^{(4)} R(\hat{g}) - \frac{1}{4} M_p^2 m^2 \int d^4x \sqrt{-\hat{g}} \mathcal{U}(\hat{g}^{-1} f) . \quad (2.1)$$

Note that by definition $\hat{g}^{\mu\nu}$ and $f_{\mu\nu}$ transform under general diffeomorphism transformations $x'^\mu = x'^\mu(x)$ as

$$\hat{g}'^{\mu\nu}(x') = \hat{g}^{\rho\sigma}(x) \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} , \quad f'_{\mu\nu}(x') = f_{\rho\sigma}(x) \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} . \quad (2.2)$$

⁴For some works, see [44, 45, 46].

Now the requirement that the combination $\hat{g}^{-1}f$ has to be diffeomorphism invariant implies that the potential \mathcal{U} has to contain the trace over space-time indices. Further, it is convenient to parameterize the tensor $f_{\mu\nu}$ using four scalar fields ϕ^A and some fixed auxiliary metric $\bar{f}_{\mu\nu}(\phi)$ so that

$$f_{\mu\nu} = \partial_\mu \phi^A \partial_\nu \phi^B \bar{f}_{AB}(\phi) , \quad (2.3)$$

where the metric f_{AB} is invariant under diffeomorphism transformation $x'^\mu = x^\mu(x')$ which however transforms as a tensor under reparametrizations of ϕ^A . In what follows we consider $\bar{f}_{AB} = \eta_{AB}$, where $\eta_{AB} = \text{diag}(-1, 1, 1, 1)$.

The fundamental ingredient of the non-linear massive gravity is the potential term. The most general forms of this potential were derived in [7, 9]. Let us consider the minimal form of the potential introduced in [9]

$$\begin{aligned} \mathcal{U}(g, H) &= -4 \left(\langle \mathcal{K} \rangle^2 - \langle \mathcal{K}^2 \rangle \right) = \\ &= -4 \left(\sum_{n \geq 1} d_n \langle H^n \rangle \right)^2 - 8 \sum_{n \geq 2} d_n \langle H^n \rangle , \end{aligned} \quad (2.4)$$

where we now have

$$\begin{aligned} H_{\mu\nu} &= \hat{g}_{\mu\nu} - \partial_\mu \phi^A \partial_\nu \phi^B \eta_{AB} , \quad H_\nu^\mu = \hat{g}^{\mu\alpha} H_{\alpha\nu} , \\ \mathcal{K}_\nu^\mu &= \delta_\nu^\mu - \sqrt{\delta_\nu^\mu - H_\nu^\mu} = - \sum_{n=1}^{\infty} d_n (H^n)_\nu^\mu , \quad d_n = \frac{(2n)!}{(1-2n)(n!)^2 4^n} . \end{aligned} \quad (2.5)$$

and where $(H^n)_\nu^\mu = H_{\alpha_1}^\mu H_{\alpha_2}^{\alpha_1} \dots H_{\nu}^{\alpha_{n-1}}$.

As was observed in [13] in order to find the Hamiltonian formulation of given action it is more convenient to consider the potential term written as the trace over Lorentz indices rather than the curved space ones. Explicitly, we have

$$\mathcal{U} = -4(\langle \hat{\mathcal{K}} \rangle^2 - \langle \hat{\mathcal{K}}^2 \rangle) = -4(\hat{\mathcal{K}}_A^A)^2 + 4\hat{\mathcal{K}}_B^A \hat{\mathcal{K}}_A^B , \quad (2.6)$$

where we defined

$$\hat{\mathcal{K}}_B^A = \delta_B^A - \sqrt{\delta_B^A - \mathbf{A}_B^A} . \quad (2.7)$$

Even if it is possible to perform the Hamiltonian analysis for general form of the potential we now consider following potential term

$$\mathcal{L}_{matt} = -2M_p^2 m^2 \sqrt{g} N \text{Tr} \sqrt{\mathbf{A}} . \quad (2.8)$$

Following [11, 12] we introduce auxiliary fields Φ_B^A and rewrite the potential term into the form

$$\mathcal{L}_{matt} = -M_p^2 m^2 \sqrt{g} N (\Phi_B^A + (\Phi^{-1})_B^A \mathbf{A}_A^B) , \quad \mathbf{A}_A^B = \hat{g}^{\mu\nu} \partial_\mu \phi^B \partial_\nu \phi_A . \quad (2.9)$$

Note that we have to presume an existence of the inverse matrix $(\Phi^{-1})_B^A$. Then in order to see the equivalence between (2.8) and (2.9) let us perform the variation of (2.9) with respect to Φ_B^A

$$\delta_B^A - (\Phi^{-1})_C^A \mathbf{A}_D^C (\Phi^{-1})_B^D = 0 \quad (2.10)$$

that implies

$$\Phi_B^A \Phi_C^B = \mathbf{A}_C^A \Rightarrow \Phi_B^A = \sqrt{\mathbf{A}_B^A} . \quad (2.11)$$

Then inserting (2.11) into (2.9) we obtain (2.8) and hence an equivalence between these two formulations is established.

Now we are ready to proceed to the Hamiltonian formalism of given theory. Explicitly, we use 3 + 1 notation [39]⁵ and write the four dimensional metric components as

$$\begin{aligned} \hat{g}_{00} &= -N^2 + N_i g^{ij} N_j , \quad \hat{g}_{0i} = N_i , \quad \hat{g}_{ij} = g_{ij} , \\ \hat{g}^{00} &= -\frac{1}{N^2} , \quad \hat{g}^{0i} = \frac{N^i}{N^2} , \quad \hat{g}^{ij} = g^{ij} - \frac{N^i N^j}{N^2} . \end{aligned} \quad (2.12)$$

Note also that 4-dimensional scalar curvature has following decomposition

$${}^{(4)}R = K_{ij} \mathcal{G}^{ijkl} K_{kl} + {}^{(3)}R , \quad (2.13)$$

where ${}^{(3)}R$ is three-dimensional spatial curvature, K_{ij} is extrinsic curvature defined as

$$K_{ij} = \frac{1}{2N} (\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i) , \quad (2.14)$$

where ∇_i is covariant derivative built from the metric components g_{ij} . Note also that \mathcal{G}^{ijkl} is de Witt metric defined as

$$\mathcal{G}^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - g^{ij} g^{kl} , \quad \mathcal{G}_{ijkl} = \frac{1}{2} (g_{ik} g_{jl} + g_{il} g_{jk}) - \frac{1}{2} g_{ij} g_{kl} . \quad (2.15)$$

Finally note that in (2.13) we omitted terms proportional to the covariant derivatives which induce the boundary terms that vanish for suitable chosen boundary conditions. Then in 3 + 1 formalism \mathbf{A}_B^A takes the form

$$\mathbf{A}_B^A = -\nabla_n \phi^A \nabla_n \phi_B + g^{ij} \partial_i \phi^A \partial_j \phi_B , \quad \nabla_n \phi^A = \frac{1}{N} (\partial_t \phi^A - N^i \partial_i \phi^A) . \quad (2.16)$$

Using this notation we rewrite the mass term (2.9) into the form

$$\begin{aligned} \mathcal{L}_{matt} &= -M_p^2 m^2 \sqrt{g} N (\Phi^{AB} \eta_{BA} - (\Phi^{-1})_{AB} \nabla_n \phi^B \nabla_n \phi^A + (\Phi^{-1})_{AB} V^{BA}) , \\ V^{BA} &= g^{ij} \partial_i \phi^B \partial_j \phi^A , \quad \Phi_{AB} = \eta_{AC} \Phi_A^C . \end{aligned} \quad (2.17)$$

The Hamiltonian analysis of the gravity part of the action is well known. Explicitly, the momenta conjugate to N, N^i are the primary constraints of the theory

$$\pi_N(\mathbf{x}) \approx 0 , \quad \pi_i(\mathbf{x}) \approx 0 \quad (2.18)$$

⁵For review, see [40].

while the Hamiltonian takes the form

$$\begin{aligned}
\mathcal{H}^{GR} &= N\mathcal{H}_T^{GR} + N^i\mathcal{H}_i^{GR} , \\
\mathcal{H}_T^{GR} &= \frac{1}{\sqrt{g}M_p^2}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - M_p^2\sqrt{g}^{(3)}R , \\
\mathcal{H}_i^{GR} &= -2g_{ik}\nabla_j\pi^{jk} ,
\end{aligned} \tag{2.19}$$

where π^{ij} are momenta conjugate to g_{ij} with following non-zero Poisson brackets

$$\{g_{ij}(\mathbf{x}), \pi^{kl}(\mathbf{y})\} = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)\delta(\mathbf{x} - \mathbf{y}) . \tag{2.20}$$

Finally note that π_N, π_i have following Poisson brackets with N, N^i

$$\{N(\mathbf{x}), \pi_N(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}) , \quad \{N^i(\mathbf{x}), \pi_j(\mathbf{y})\} = \delta_j^i\delta(\mathbf{x} - \mathbf{y}) . \tag{2.21}$$

Now we proceed to the Hamiltonian analysis of the scalar field part of the action. The momenta conjugate Φ^{AB} are the primary constraints

$$P_{AB} = \frac{\delta\mathcal{L}_{matt}}{\delta\partial_t\Phi^{AB}} \approx 0 , \tag{2.22}$$

while the momenta conjugate to ϕ^A are related to the time derivative of ϕ^A by

$$p_A = \frac{\delta\mathcal{L}_{matt}}{\delta\partial_t\phi^A} = 2M_p^2m^2\sqrt{g}(\Phi^{-1})_{AB}\nabla_n\phi^B . \tag{2.23}$$

Note that these canonical variables have following non-zero Poisson brackets

$$\{\Phi^{AB}(\mathbf{x}), P_{CD}(\mathbf{y})\} = \frac{1}{2}(\delta_C^A\delta_D^B + \delta_D^A\delta_C^B)\delta(\mathbf{x} - \mathbf{y}) , \quad \{\phi^A(\mathbf{x}), p_B(\mathbf{y})\} = \delta_B^A\delta(\mathbf{x} - \mathbf{y}) . \tag{2.24}$$

Using these variables we find that the Hamiltonian for the scalar fields takes the form

$$\begin{aligned}
\mathcal{H}^{sc} &= N\mathcal{H}_T^{sc} + N^i\mathcal{H}_i^{sc} + \Omega^{AB}P_{BA} , \quad \mathcal{H}_i^{sc} = p_A\partial_i\phi^A , \\
\mathcal{H}_T^{sc} &= \frac{1}{4\sqrt{g}M_p^2m^2}\Phi^{AB}p_Bp_A + M_p^2m^2\sqrt{g}[\Phi^{AB}\eta_{BA} + (\Phi^{-1})_{AB}V^{BA}] .
\end{aligned} \tag{2.25}$$

Now we analyze the requirement of the preservation of the primary constraints during the time evolution of the system. Note that the Hamiltonian is equal to

$$H = \int d^3\mathbf{x}(N\mathcal{H}_T + N^i\mathcal{H}_i + \Omega^{AB}P_{BA} + u^N\pi_N + u^i\pi_i) , \tag{2.26}$$

where $\mathcal{H}_T = \mathcal{H}_T^{GR} + \mathcal{H}_T^{sc}$, $\mathcal{H}_i = \mathcal{H}_i^{GR} + \mathcal{H}_i^{sc}$ and where Ω^{AB}, u^N and u^i are Lagrange multipliers corresponding to the primary constraints $P_{AB} \approx 0, \pi^N \approx 0$ and $\pi_i \approx 0$.

As usual the requirement of the preservation of the primary constraints $\pi_N \approx 0, \pi_i \approx 0$ implies following secondary constraints

$$\mathcal{H}_T \approx 0 , \quad \mathcal{H}_i \approx 0 . \tag{2.27}$$

On the other hand the requirement of the preservation of the primary constraints $P_{AB} \approx 0$ implies following secondary ones

$$\begin{aligned} \partial_t P_{AB} &= \{P_{AB}, H\} = -\frac{N}{4M_p^2 m^2 \sqrt{g}} p_A p_B + \\ &+ NM_p^2 m^2 \sqrt{g} [-\eta_{BA} + (\Phi^{-1})_{AC} V^{CD} (\Phi^{-1})_{DB}] \equiv N\Psi_{AB} \approx 0 \end{aligned} \quad (2.28)$$

using

$$\begin{aligned} \{P_{AB}(\mathbf{x}), (\Phi^{-1})_{CD}(\mathbf{y})\} &= -(\Phi^{-1})_{CK}(\mathbf{y}) \{P_{AB}(\mathbf{x}), \Phi^{KL}(\mathbf{y})\} (\Phi^{-1})_{LD}(\mathbf{y}) = \\ &= \frac{1}{2} ((\Phi^{-1})_{AC} (\Phi^{-1})_{BD} + (\Phi^{-1})_{AD} (\Phi^{-1})_{BC}) (\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) . \end{aligned} \quad (2.29)$$

For further purposes it is useful to know the Poisson bracket between P_{AB} and Ψ_{CD}

$$\begin{aligned} \{P_{AB}(\mathbf{x}), \Psi_{CD}(\mathbf{y})\} &= \frac{1}{2} M_p^2 m^2 \sqrt{g} [((\Phi^{-1})_{AC} (\Phi^{-1})_{BE} + (\Phi^{-1})_{AE} (\Phi^{-1})_{BC}) V^{EF} (\Phi^{-1})_{FD} + \\ &+ (\Phi^{-1})_{CE} V^{EF} ((\Phi^{-1})_{AF} (\Phi^{-1})_{BD} + (\Phi^{-1})_{AD} (\Phi^{-1})_{BF})] \delta(\mathbf{x} - \mathbf{y}) \equiv \\ &\equiv \Delta_{AB,CD}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) . \end{aligned} \quad (2.30)$$

Note that there are non-trivial Poisson brackets $\{\Psi_{AB}(\mathbf{x}), \Psi_{CD}(\mathbf{y})\}$ and $\{\Psi_{AB}(\mathbf{x}), \mathcal{H}_T(\mathbf{y})\} \equiv \Delta_{AB}(\mathbf{x}, \mathbf{y})$. Fortunately their explicit forms will not be needed. Finally note that it is useful to write the total Hamiltonian in the form

$$\begin{aligned} H_T &= \int d^3 \mathbf{x} (N \mathcal{H}_T + \Omega^{AB} P_{BA} + \Gamma^{AB} \Psi_{BA}) + \mathbf{T}_S(N^i) , \\ \mathbf{T}_S(N^i) &= \int d^3 \mathbf{x} N^i (\mathcal{H}_i + \partial_i \Phi^{AB} P_{BA}) . \end{aligned} \quad (2.31)$$

Using the smeared form of the diffeomorphism constraint $\mathbf{T}_S(N^i)$ we find the Poisson bracket

$$\{\mathbf{T}_S(N^i), \Sigma_{AB}\} = -\partial_i N^i \Sigma_{AB} - N^i \partial_i \Sigma_{AB} . \quad (2.32)$$

Then it is easy to determine the time evolution of all constraints. Note also that the Poisson brackets between smeared form of the Hamiltonian and diffeomorphism constraints take the standard form

$$\begin{aligned} \{\mathbf{T}_T(N), \mathbf{T}_T(M)\} &= \mathbf{T}_S((N \partial_j M - M \partial_j N) g^{ji}) , \\ \{\mathbf{T}_S(N^i), \mathbf{T}_S(M^j)\} &= \mathbf{T}_S(N^j \partial_j M^i - M^j \partial_j N^i) , \\ \{\mathbf{T}_S(N^i), \mathbf{T}_T(N)\} &= \mathbf{T}_T(\partial_k N N^k) . \end{aligned} \quad (2.33)$$

In case of the constraints P_{AB} and Ψ_{AB} we obtain

$$\begin{aligned}\partial_t P_{AB} &= \{P_{AB}, H_T\} \approx \Delta_{AB,CD} \Gamma^{CD} = 0, \\ \partial_t \Psi_{AB} &\approx \int d^3 \mathbf{x} (\{\Psi_{AB}, N \mathcal{H}_T(\mathbf{x})\} + \Gamma^{CD} \{\Psi_{AB}, \Psi_{CD}(\mathbf{x})\} + \Omega^{CD} \{\Psi_{AB}, P_{CD}(\mathbf{x})\}) = 0.\end{aligned}\tag{2.34}$$

The crucial point is to find the solution of the equation

$$\Delta_{AB,CD} \Gamma^{CD} = 0.\tag{2.35}$$

To find such a solution we consider an ansatz

$$\Gamma^{CD} = u^C u^D,\tag{2.36}$$

where u^C are 4-dimensional vectors. Then the condition (2.35) implies

$$(\Phi^{-1})_{AB} u^B = 0.\tag{2.37}$$

By presumption Φ is non-singular matrix so that $\det(\Phi^{-1}) \neq 0$. Then clearly the only solution of the equations (2.37) is $u^A = 0$ so that $\Gamma^{CD} = 0$. However there is also another possibility. In fact, let us consider the ansatz

$$\Gamma^{CD} = \Phi^{CA} u_A \Phi^{DB} u_B\tag{2.38}$$

so that the condition $\Delta_{AB,CD} \Gamma^{CD} = 0$ is equivalent to the equation

$$V^{AB} u_B = 0.\tag{2.39}$$

For generic point of the phase space we can demand that $\det V^{AB} \neq 0$ and hence the above condition is solved by $u_A = 0$. However we should stress that the case $\det V^{AB} = 0$ is also very important and deserves separate treatment that will be performed in the next section.

Returning to the non-degenerative case we find that thanks to the solution $\Gamma^{AB} = 0$ the second equation in (2.34) takes the form

$$\int d^3 \mathbf{x} N(\mathbf{x}) \Delta_{AB}(\mathbf{x}, \mathbf{y}) - \Omega^{CD}(\mathbf{y}) \Delta_{CD,AB}(\mathbf{y}) = 0.\tag{2.40}$$

Again, due to the properties of the matrix $\Delta_{AB,CD}$ we find that it has unique solution. As a result the Lagrange multipliers Γ^{CD} and Ω^{CD} are determined while $N(\mathbf{x})$ is free which is the reflection of the fact that the temporal diffeomorphism is preserved. To see this from different point of view note that the constraint \mathcal{H}_T is preserved during the time evolution of the system since

$$\partial_t \mathcal{H}_T = \{\mathcal{H}_T, H_T\} \approx \int d^3 \mathbf{x} \Gamma^{AB}(\mathbf{x}) \{\mathcal{H}_T, \Psi_{AB}(\mathbf{x})\} = 0\tag{2.41}$$

due to the fact that $\Gamma^{AB} = 0$. As a result we have ten second class constraints $P_{AB} \approx 0, \Psi_{AB} \approx 0$. Solving the constraint $\Psi_{AB} = 0$ we find

$$(\Phi^{-1})_{AB} = \sqrt{\left(\frac{p_A p_C}{4(M_p^2 m^2 \sqrt{g})^2} + \eta_{AC} \right) V^{CD} (V^{-1})_{DB}}. \quad (2.42)$$

Then inserting this result into the scalar field part of the Hamiltonian constraint we obtain

$$\mathcal{H}_T^{sc} = 2M_p^2 m^2 \sqrt{g} \sqrt{\left(\frac{p_A p_C}{4(M_p^2 m^2 \sqrt{g})^2} + \eta_{AC} \right) V^{CA}}. \quad (2.43)$$

This form of the Hamiltonian constraint agrees with the Hamiltonian constraint found recently in [13].

Let us outline our results. We analyzed the model of non-linear massive gravity with Stückelberg fields and with also auxiliary fields that allow to consider theory without the square root structure. We shown that the structure of the constraints has the standard form corresponding to the diffeomorphism invariant theory without any new additional constraint that could eliminate one scalar degree of freedom.

However the analysis presented in this paper was based on one important presumption which is the requirement that $\det V^{AB} \neq 0$. In fact, when we try to impose the unitary gauge

$$\mathcal{G}^A : \phi^A - x^\mu \delta_\mu^A = 0 \quad (2.44)$$

we find that $V^{00} = V^{0i} = 0$ for $i = 1, 2, 3$ and consequently $\det V^{AB} = 0$. In other words the unitary gauge (2.44) cannot be imposed at the generic points of the phase space where $\det V^{AB} \neq 0$. Clearly the analysis of the system where $\det V^{AB} = 0$ deserves separate treatment.

3. Degenerative Case $\det V^{AB} = 0$

As we argued above the non-linear massive theory has an interesting property that its phase space structure changes when we move over the phase space. To see this note that for the subspace of the phase space characterized by the condition $\det V^{AB} = 0$ we find that the matrix $\Delta_{AB,CD}$ possesses the zero modes. Explicitly, for the ansatz

$$\Gamma^{AB} = \Phi^{AC} u_C \Phi^{BD} u_D = \Gamma^{BA} \quad (3.1)$$

we obtain

$$\Delta_{AB,CD} \Gamma^{CD} = M_p^2 m^2 \sqrt{g} [u_A (\Phi^{-1})_{BE} V^{EF} u_F + (\Phi^{-1})_{AE} V^{EF} u_F u_B] \quad (3.2)$$

and hence this equation is equal to zero for u_A obeying the equation

$$V^{AB} u_B = 0. \quad (3.3)$$

Clearly the number of zero modes is determined by the rank of the matrix V^{AB} . We restrict ourselves to the case when the number of zero modes is equal to one and we denote corresponding vector as u_A^0 . Let us then consider following linear combination

$$P^M = \Phi^{AC} u_C^0 \Phi^{BD} u_D^0 P_{AB} . \quad (3.4)$$

Then $\{P^M, \Sigma_{AB}\} \approx 0$ and using (3.3) we find

$$\partial_t P^M = \{P^M, H_T\} \approx 0 \quad (3.5)$$

and hence P^M is the first class constraint of the theory. In other words from 10 constraints P^{AB} we find one the first class constraint and 9 the second class constraints. Note that these second class constraints are preserved during the time evolution of the system on condition when corresponding Lagrange multipliers are zero. Let us consider the constraint

$$\Psi_M = \Psi_{AB} \Phi^{AC} u_C^0 \Phi^{BD} u_D^0 . \quad (3.6)$$

Then the time evolution of given constraint is equal to

$$\begin{aligned} \partial_t \Psi_M &\approx \{\Psi_M, H_T\} = \int d^3 \mathbf{x} (N(\mathbf{x}) \{\Psi_M, \mathcal{H}_T(\mathbf{x})\} + \\ &+ \{\Psi_M, \Psi_{AB}(\mathbf{x})\} \Gamma^{AB}(\mathbf{x}) + \{\Psi_M, P_{AB}(\mathbf{x})\} \Omega^{AB}(\mathbf{x})) = \\ &= \int d^3 \mathbf{x} (N \{\Psi_M, \mathcal{H}_T\}) = 0 \end{aligned} \quad (3.7)$$

using the fact that $\{\Psi_M(\mathbf{y}), \Psi_{AB}(\mathbf{x})\} \Gamma^{AB}(\mathbf{x}) = \{\Psi_M(\mathbf{y}), \Psi_M(\mathbf{x})\} = 0$ and also using the fact that

$$\{\Psi_M(\mathbf{x}), P_{AB}(\mathbf{y})\} \approx u_C^0 \Phi^{CE} u_D^0 \Phi^{DF} \{\Psi_{EF}(\mathbf{x}), P_{AB}(\mathbf{y})\} = 0 . \quad (3.8)$$

To proceed further we have to calculate the Poisson bracket between $\Psi_M = \frac{1}{4M_p^2 m^2 \sqrt{g}} (u_A^0 \Phi^{AB} p_B)^2 + M_p^2 m^2 \sqrt{g} \eta_{AB} \Phi^{AC} \Phi^{BD} u_C^0 u_D^0$ and \mathcal{H}_T

$$\begin{aligned} \{\Psi_M(\mathbf{x}), \mathcal{H}_T(\mathbf{y})\} &= \\ &= -\frac{1}{8M_p^4 m^4 (\sqrt{g})^2} g_{ij} \pi^{ji} (u_A^0 \Phi^{AB} p_B)^2 \delta(\mathbf{x} - \mathbf{y}) + \frac{1}{2} g_{ij} \pi^{ji} \eta_{AB} \Phi^{AC} u_C^0 \Phi^{BD} u_D^0 \delta(\mathbf{x} - \mathbf{y}) - \\ &- (u_A^0 \Phi^{AB} p_B) u_C^0 g^{ij} \partial_i \phi^C \partial_j \delta(\mathbf{x} - \mathbf{y}) - \frac{1}{\sqrt{g}} (u_A^0 \Phi^{AB} p_B) u_D^0 \Phi^{DC} \partial_i [\sqrt{g} g^{ij} \partial_j \phi^E \Phi_{EC}] \delta(\mathbf{x} - \mathbf{y}) \end{aligned} \quad (3.9)$$

using

$$\{\sqrt{g}(\mathbf{x}), \mathcal{H}_T^{GR}(\mathbf{y})\} = \frac{1}{2M_p^2 m^2} g_{ij} \pi^{ji}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) . \quad (3.10)$$

Since (3.9) is non-zero we obtain that the equation (3.7) implies that the Lagrange multiplier N should be equal to zero.

However at this place we should stress one important point. Let us imagine that instead of the Hamiltonian constraint we have Hamiltonian that does not vanish on the

constraint surface. In this case the equation (3.7) would imply an additional constraint that would be the second class constraint together with Ψ_M . However the crucial point is that the original Hamiltonian is the linear combination of the constraints and consequently the non-zero Poisson bracket (3.9) determines corresponding Lagrange multiplier rather than an additional constraint. As a result Ψ_M and \mathcal{H}_T are the second class constraints. We also have 18 the second class constraints from original Ψ_{AB} and P_{AB} . These constraints together with the first class constraint P_M eliminates P_{AB} and Φ^{AB} while \mathcal{H}_T and Ψ_M eliminates one scalar degree of freedom. The remaining 3 scalar degrees of freedom can be eliminated by gauge fixing \mathcal{H}_i . As a result we obtain that the physical degrees of freedom are 6 metric components g_{ij} .

At this place we should stress that we have to be very careful with such conclusion. The problem is that it is not completely clear to us how to proceed with the system that degenerates at the subspace of the phase space. More precisely, it is not clear whether we should interpret the condition $\det V^{AB} = 0$ as the primary constraint of the theory. However the presence of such a constraint would completely change the constraint structure of given theory. We leave the complete analysis of this situation for future.

In any case it is very interesting problem that the Poisson brackets between constraints depends on the phase space variables. It is possible that this fact could provide an explanation why the gauge fixed form of the theory seems to be able to eliminate one additional scalar mode [37, 38] while in the case of the manifestly diffeomorphism invariant theory defined at the generic point of the phase space such an additional constraint is lacking.

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